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NONLINEAR ADAPTIVE CONTROL SYSTEMS WITH STORAGE

by

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ABSTRACT

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This paper considers adaptive control systems which have nonlinear controllers with storage.Staircase techniques have been used to derive the optimizing equations for calculating the optimum parameters of the compensator for the least mean square error between the actual output and the desired output in terms of the statistical properties of the input signal and the plant dynamics.

An example of an input-adaptive system has been calculated showing the superior performance of a power-series controller with storage over the optimum linear controller, as well as the optimum power-series controller without storage.

Author

INTRODUCTION

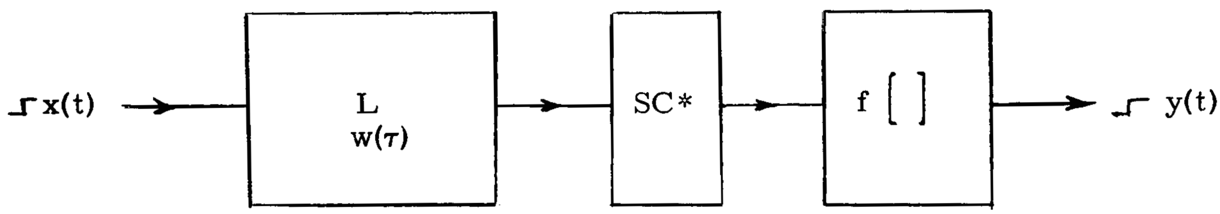
In a recent report[1] the theory of adaptive systems with nonlinear compensation was developed by using staircase techniques. The conventional linear compensator, with variable coefficients, normally used for modification in an adaptive system was replaced by an instantaneous (no-storage) nonlinear compensator. Optimum coefficients for this nonlinear compensator were calculated from the statistical parameters of the input function and the plant dynamics, the criterian for optimization being the smallest mean square error between the actual output and the desired output. An example was calculated to illustrate that in many cases even a simple power-series device could give a smaller mean square error than the best linear compensator.

The object of this report is to extend the previous work to include the case where the compensator is a nonlinear device with storage. The validity of this extension is based on the Wiener-Bose theory[2] that a nonlinear device with storage is equivalent to the cascade combination of an instantaneous nonlinear device and a linear device with storage elements. Wiener showed that any system can be regarded as a computer which performs a transformation on the past of its input to yield the present output. For the case where the transformation is linear, the convolution integral can be used to obtain the present output from the past of the input. Hence, any linear system is characteriqed by its response to a unit impulse. Similarly any nonlinear system (with finite settling time) can be characterized by a linear network with multiple outputs cascaded with a nonlinear network with no memory of the past. This is justifiable because the linear network serves to characterize the past of the input and the nonlinear network operates on this information to yield the present output.

In this paper, as in the previous, it is assumed that the input data are sampled at regular intervals, the sampling rate being fixed in accordance with Shannon's sampling theorem, and then converted into a staircase function by using a zero-order hold. This assumption is necessary in order to overcome the mathematical intractability of the equations arising out of the optimization of nonlinear systems subject to continuous random inputs[5]. The dynamics of the plant are considered as known in terms of its staircase P-response, and although it is not necessary, for the sake of simplicity it has been assumed that the plant is time-invariant. The method developed is also applicable, with a slight modification, for the case where the plant dynamics are varying slowly with time.

NONLINEAR STAIRCASE SYSTEMS WITH STORAGE

The general nonlinear staircase system with storage may be represented by the block diagram shown in Figure 1, where L is a linear system of weighting function $w(\tau)$ and $f[\ ]$ is a no-memory nonlinear device.



*SC = Sampler and Clamp

Figure 1. A General Nonlinear Staircase System with Storage

The output of the system at the k th sampling instant may be expressed as[5]

$$
\begin{aligned}
y(kT) &= f\left[x_o u_k + x_1 u_{k-1} + \cdots + x_k u_o\right] \\
&= f \sum_{s=0}^{k} u_s x_{k-s} \qquad (2.01)
\end{aligned}
$$

where

$x_r = x(rT)$

$u_r = u(rT)$

= staircase P-response of L at the k th sampling instant.

In the particular case of a first-order system, belonging to Class N_1 of Zadeh[6,7], it is permissible to represent it by the block diagram shown in Figure 2. For this case, the staircase output at the k th sampling instant is given by

$$y_k = \sum_{r=0}^{k} f(x_r)\ u_{k-r} \qquad (2.02)$$

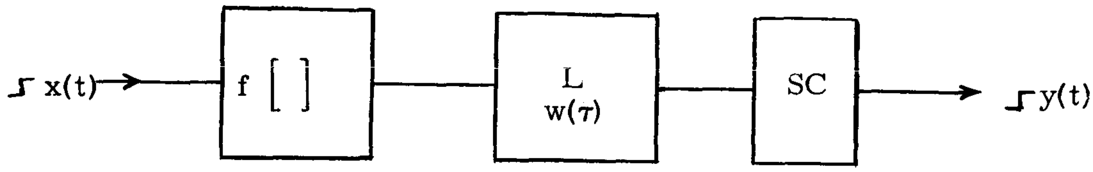


Figure 2. Storage Nonlinear Staircase System of First Order

It may be noted that in deriving Equations (2.01) and (2.02) it has been assumed that the linear system is physically realizable, that is, for L the staircase P-response ordinate u_r does not exist for r less than 0.

As the optimizing equations for higher order systems are considerably more involved, in this work only systems of the first order will be considered.

OPTIMIZATION OF ADAPTIVE SYSTEMS HAVING FIRST-ORDER NONLINEAR CONTROLLERS WITH STORAGE

The block diagram of a model-reference type adaptive control system is shown in Figure 3. The "controller" performs a suitable transformation on the input signal so that the output of the known plant corresponds as closely to the desired output as possible. In order that this may be done a computer is required, which would use the statistical properties of the input, the desired output and the plant dynamics to calculate the optimum controller for a given index of performance.

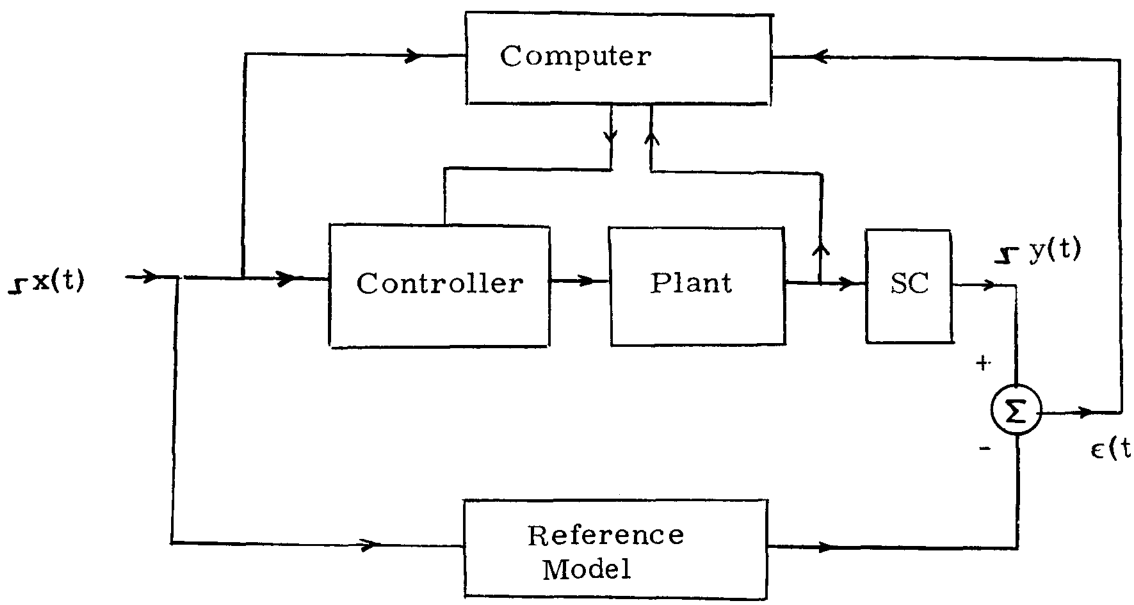


Figure 3. A Model Reference Adaptive Control System

To render the problem more practicable, the form of the controller is first assumed, and then its parameters are calculated. For instance, one may consider a linear controller, and calculate its weighting function, $w(\tau)$, which will give the least mean-square error between the desired output and the actual output. It was shown[1] that the staircase P-response ordinates, u_r, of the optimum linear controller can be calculated by solving the following set of linear simultaneous equations:

$$\begin{aligned}
\alpha_o u_o + \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_N u_N &= \beta_o \\
\alpha_1 u_o + \alpha_o u_1 + \alpha_1 u_2 + \cdots + \alpha_{N-1} u_N &= \beta_1 \\
\alpha_2 u_o + \alpha_1 u_1 + \alpha_o u_2 + \cdots + \alpha_{N-2} u_N &= \beta_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
\alpha_N u_o + \alpha_{N-1} u_1 + \alpha_{N-2} u_2 + \cdots + \alpha_o u_N &= \beta_N
\end{aligned} \tag{3.01}$$

where

$$\alpha_k = \sum_{r=0}^{N} \phi_{\gamma\gamma}(kT)\, \phi_{xx}(\overline{k+r}\, T) \tag{3.02}$$

$$\beta_k = \sum_{r=0}^{N} \gamma(rT)\, \phi_{xz}(\overline{k+r}\; T) \tag{3.03}$$

$\gamma(rT) =$ The staircase P-response ordinate of the plant at $t = rT$ (3.04)

$$\phi_{\gamma\gamma}(rT) = \sum_{k=0}^{N-r} V(kT)\;\; \gamma(\overline{k+r}\; T) \tag{3.05}$$

$$\phi_{xx}(rT) = \frac{1}{N-1} \sum_{k=0}^{N-r} x(kT)\;\; x(\overline{k+r}\; T) \tag{3.06}$$

and

$$\phi_{xz}(rT) = \frac{1}{N-r+1} \sum_{k=0}^{N-r} x(kT)\;\; z(k+rT) \tag{3.07}$$

and

$z(rT) =$ the desired output at $t = rT$. (3.08)

On the other hand, if the controller is a no-storage linear device with input-output relationship characterized by

$$y\,(rT) = \sum_{k=0}^{M} a_k\, f_k\left[x(rT)\right] \tag{3.09}$$

where $f_k[\;]$ is a known nonlinear functional, and a_k is an unknown coefficient to be calculated, it has been shown[1] that a_k can be obtained by solving the following set of linear simultaneous equations:

$$\sum_{m=0}^{M} a_m \sum_{r=0}^{N} \sum_{s=0}^{N} \gamma_r \gamma_s\, \phi_{mj}\,(f;\; \overline{r-s}\;\; T)$$

$$= \sum_{r=0}^{N} \gamma_r\, \phi_{jz}\,(f;\, rT) \quad \text{for } j = 0, 1, 2, \cdots M, \tag{3.10}$$

where the nonlinear correlation ordinates $\phi_{mj}\,(f;\; \overline{r-s}\;\; T)$ and $\phi_{jz}\,(f;\, rT)$ are defined as below:

$$\phi_{mj}\,(f;\; \overline{r-s}\;\; T) = \frac{1}{N - s + 1} \sum_{p=0}^{N-s} f_m\left[x_p\right]\; f_j\left[x_{p+s-r}\right]$$

$$\text{for } s > r$$

$$= \frac{1}{N - r + 1} \sum_{p=0}^{N-r} f_m\left[x_p\right]\; f_j\left[x_{p+r-s}\right]$$

$$\text{for } s < r. \tag{3.11}$$

and

$$\phi_{jz}\,(f; rT) = \frac{1}{N - r + 1} \sum_{p=0}^{N-r} f_j\left[x_p\right]\; z_{p+r} \tag{3.12}$$

An important subclass of no-storage nonlinear devices characterized by Equation (3.09) is the instantaneous power-series device the output and input for which are related by

$$y_r = \sum_{k=0}^{M} a_k x_r^k \tag{3.13}$$

The optimum values of the coefficients a_k which will give the least mean square error between the desired output and actual output are obtained by solving the following equations:

$$\left.\begin{array}{l} \alpha_{11}a_1 + \alpha_{12}a_2 + \cdots + \alpha_{1M}a_M = \beta_1 \\ \alpha_{21}a_1 + \alpha_{22}a_2 + \cdots + \alpha_{2M}a_M = \beta_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_{M1}a_1 + \alpha_{M2}a_2 + \cdots + \alpha_{MM}a_M = \beta_M \end{array}\right\} \tag{3.14}$$

where

$$\alpha_{jk} = \sum_{r=0}^{N} \sum_{s=0}^{N} \gamma_r \gamma_s \, \phi_{x^j x^s} (\overline{r-s}\ T) \tag{3.15}$$

and

$$\beta_k = \sum_{r=0}^{N} \gamma_r \, \phi_{x^k z} (rT) \tag{3.16}$$

The nonlinear correlation functions in Equations (3.15) and (3.16) are defined as

$$\phi_{x^j x^k} (rT) = \frac{1}{N - r + 1} \sum_{p=0}^{N-r} x_p^j \, x_{p+r}^k \tag{3.17}$$

and

$$\phi_{x^k z} (rT) = \frac{1}{N - r + 1} \sum_{p=0}^{N-r} x_p^k \quad z_{p+r} \tag{3.18}$$

If the controller is assumed to be nonlinear with storage, it can be represented as the cascade combination of a linear system L, of weighting function $w(\tau)$ and an instantaneous nonlinear device, as shown in Figure 1. On the other hand a nonlinear system of the first order may be considered as the cascade combination shown in Figure 2. Assuming that the controller is of the latter type, one would have to calculate the staircase P-response ordinates of the linear system L, as well as the coefficients, a_r, of the instantaneous nonlinear system which will together yield the minimum mean square error between the actual output and the desired output. In practice, however, this approach is not very fruitful as the optimizing equations get very involved.

An alternative approach, which consists of optimization in two steps, is much more practicable. The staircase P-response ordinates of the optimum linear controller are first calculated; and it is assumed the optimum nonlinear controller would consist of a cascade combination of an instantaneous nonlinear device and this optimum linear device, with the arrangement shown in Figure 2. Hence, after staircase P-response of L has been obtained by using Equations (3.01), these may be convolved with the staircase P-response ordinates of the plant, $\gamma(kT)$. The result of the convolution, $v(kT)$, may now be used in place of $\gamma(kT)$ in Equations (3.10), to obtain the coefficients a_k of the optimum instantaneous nonlinear controller defined in Equation (3.09).

This procedure, therefore, gives a cascade combination of the optimum linear controller, preceded by a suitable instantaneous nonlinear device. It may be pointed out that, in general, the performance of this combination will be better than that of either of the components alone.

EXAMPLE OF AN INPUT-SENSING NONLINEAR ADAPTIVE SYSTEM WITH STORAGE

To compare the relative performances of the various types of controllers discussed in the previous section, an example of an input-sensing adaptive system may be taken. The plant is assumed time-invariant, and the desired output is taken as being equal to the input. The transfer function of the plant may be taken as $\frac{1}{(s+1)(s+10)}$. For this transfer function, the staircase P-response ordinates are given by

$$\gamma(0) = 0 \tag{4.01}$$

$$\gamma(mT) = \frac{1}{9}(1 - e^{-0.1})\, e^{-0.1(m-1)} - \frac{1}{90}(1 - e^{-1})\, e^{-(m-1)} \tag{4.02}$$

where the sampling interval, T=0.1 second.

The values of $\gamma(mT)$ were calculated for m = 0 to 10, and then used to calculate the plant correlation ordinates $\phi_{\gamma\gamma}(mT)$, as defined in Equation (3.05). These values are shown in the following table:

m	$\gamma(mT)$	$\phi_{\gamma\gamma}(mT)$
0	0	0.0003677154
1	0.003550058	0.0003451441
2	0.006983582	0.0003045481
3	0.007706511	0.0002591112
4	0.007483446	0.0002134002
5	0.006959068	0.0001688645
6	0.006365899	0.000125922
7	0.005785515	0.000084902
8	0.005244299	0.00004687384
9	0.004748677	0.00001525831
10	0.004298045	0

The random input to this plant is given below:

m	x(mT)
0	1.05
1	2.24
2	2.41
3	4.22
4	3.76
5	7.79
6	9.96
7	9.63
8	8.96
9	8.54
10	5.86
11	2.89
12	6.36
13	9.40
14	1.04
15	7.09
16	5.11
17	2.40
18	0.1
19	5.22
20	0.71

Using Equations (3.06) and (3.17), various linear and nonlinear correlation ordinates may be calculated for an optimum power-series controller. These are given on the following page.

m	ϕ_{xx}(mT)	ϕ_{xx^2}(mT)	ϕ_{xx^3}(mT)	ϕ_{xx^4}(mT)	ϕ_{xx^5}(mT)
0	28.05965	219.6037	1846.516	16181.13	145557.3
1	25.07729	198.1252	1702.325	15168.66	137959.0
2	24.71797	189.1206	1570.523	13569.85	120076.1
3	24.79793	180.1209	1438.085	12061.35	104435.9
4	22.69266	155.2635	1195.828	9770.167	82981.43
5	18.94288	124.2064	942.3560	7639.574	64620.73
6	17.26048	107.4726	789.5974	6237.582	51638.22
7	17.20313	97.51853	647.2070	4631.214	35085.48
8	13.17773	65.90280	410.3618	2788.240	20112.11
9	13.28832	64.52391	365.3017	2185.960	13654.41
10	13.10950	60.03158	312.8916	1680.060	9145.209

m	ϕ_{x^2x}(mT)	$\phi_{x^2x^2}$(mT)	$\phi_{x^2x^3}$(mT)	$\phi_{x^2x^4}$(mT)	$\phi_{x^2x^5}$(mT)
0	219.6037	1846.516	16181.14	145557.3	1332632.0
1	197.0056	1648.127	14558.88	131758.5	1209639.0
2	186.7973	1465.940	12281.87	106252.6	937139.6
3	179.0746	1290.391	10111.12	82926.19	701059.9
4	153.8706	983.8791	7086.670	54457.12	437911.2
5	116.3172	665.3069	4510.910	33139.45	257053.5
6	104.1281	564.8315	3704.788	26241.93	195562.1
7	105.4692	525.0310	3103.490	19568.34	129526.1
8	73.66884	302.3845	1635.747	9704.249	61131.38
9	83.44400	376.3761	1983.009	10846.79	60628.96
10	97.70171	452.4931	2319.410	12090.70	63401.66

m	ϕ_{x^3x}(mT)	$\phi_{x^3x^2}$(mT)	$\phi_{x^3x^3}$(mT)	$\phi_{x^3x^4}$(mT)	$\phi_{x^3x^5}$(mT)
0	1846.516	16181.14	145557.3	1332632.0	12353830.0
1	1695.835	14582.62	130346.9	1186774.0	10930490.0
2	1571.307	12468.74	104778.5	905899.9	7966422.0
3	1470.086	10515.82	81174.57	654186.3	5426584.0
4	1213.977	7403.524	50564.74	368442.6	2814930.0
5	896.6653	4693.088	29049.83	194284.5	1368588.0
6	796.2246	4044.293	24872.83	163319.6	1114441.0
7	769.7933	3547.440	19831.48	117166.0	714222.7
8	538.7424	1978.158	9806.000	52782.42	295692.8
9	641.9400	2795.506	14282.33	75211.07	400616.9
10	847.1926	3999.397	20446.34	10558.89	546606.1

m	ϕ_{x^4x}(mT)	$\phi_{x^4x^2}$(mT)	$\phi_{x^4x^3}$(mT)	$\phi_{x^4x^4}$(mT)	$\phi_{x^4x^5}$(mT)
0	16181.13	145557.3	1332632.0	12353830.0	115585600.0
1	15187.66	132648.8	1193013.0	10893270.0	100456000.0
2	13908.55	111268.3	937151.9	8099677.0	71082890.0
3	12846.77	91701.55	702263.3	5602791.0	45946190.0
4	10321.55	61086.71	402131.8	2817615.0	20683940.0
5	7689.591	38515.57	226280.4	1425607.0	9367056.0
6	6829.470	34137.43	205334.5	1303135.0	8480916.0
7	6083.193	26492.93	144203.3	830448.8	4895162.0
8	4482.192	15646.03	74630.6	384156.2	2025677.0
9	5458.967	23361.83	117620.0	608655.0	3173124.0
10	7728.486	37002.38	189231.7	974485.1	5024439.0

m	$\phi_{x^5 x}(mT)$	$\phi_{x^5 x^2}(mT)$	$\phi_{x^5 x^3}(mT)$	$\phi_{x^5 x^4}(mT)$	$\phi_{x^5 x^5}(mT)$
0	145557.3	1332632.0	12353830.0	115585600.0	1089188000.0
1	138851.4	1224971.0	11061110.0	101182600.0	933765900.0
2	126590.0	1020039.0	8610405.0	74415720.0	652128300.0
3	116050.4	829441.0	6331608.0	50266240.0	409715500.0
4	91326.02	529575.5	3395568.0	23113710.0	164597300.0
5	69239.48	339921.7	1945815.0	11893250.0	75271180.0
6	61997.90	311914.0	1871666.0	11750270.0	75028990.0
7	50075.86	207040.1	1106885.0	6304361.0	36696890.0
8	39419.12	134080.8	629255.7	3185603.0	16417940.0
9	48920.23	207725.1	1038414.0	5327348.0	27482100.0
10	71966.29	347839.4	1779816.0	9155225.0	47130420.0

To calculate the optimum linear controller, first the values of the α's and β's, as defined in Equations (3.02) and (3.03) are calculated. These are given below:

$\alpha_0 = 0.04542239$ $\qquad$ $\beta_0 = 1.152355$
$\alpha_1 = 0.04224973$ $\qquad$ $\beta_1 = 1.068070$
$\alpha_2 = 0.03964489$ $\qquad$ $\beta_2 = 0.9735248$
$\alpha_3 = 0.03660728$ $\qquad$ $\beta_3 = 0.8869045$
$\alpha_4 = 0.03299484$ $\qquad$ $\beta_4 = 0.8110975$
$\alpha_5 = 0.02969728$ $\qquad$ $\beta_5 = 0.7395363$
$\alpha_6 = 0.02736118$ $\qquad$ $\beta_6 = 0.6725995$
$\alpha_7 = 0.02522141$ $\qquad$ $\beta_7 = 0.6066298$
$\alpha_8 = 0.02269131$ $\qquad$ $\beta_8 = 0.5523519$
$\alpha_9 = 0.02127776$ $\qquad$ $\beta_9 = 0.4910136$
$\alpha_{10} = 0.01943661$ $\qquad$ $\beta_{10} = 0.4187140$

Using these values of α's and β's, the staircase P-response ordinates of the optimum linear controller are calculated through Equations (3.01). These are given on the following page.

$u_0 = 26.93844$
$u_1 = 4.234751$
$u_2 = -4.883514$
$u_3 = -4.199415$
$u_4 = 1.128078$
$u_5 = 3.567722$
$u_6 = 0.6684761$
$u_7 = -3.806203$
$u_8 = 4.625687$
$u_9 = 1.321865$
$u_{10} = -5.219115$

The mean square error between the desired output and the actual output of the linear controller alone is used is given by

$$\overline{\epsilon^2} = \sum_{r=0}^{10} u_r \beta_r - \phi_{xx}(0) = 6.11438. \tag{4.03}$$

To calculate the nonlinear controller which should be connected in cascade with this optimum linear controller, the first step is to obtain the convolution of the staircase P-response of the linear controller and the staircase P-response of the plant using the relationship

$$v_k = \sum_{r=0}^{k} u_r \gamma_{k+r} . \tag{4.04}$$

These values of v_k are used to calculate the correlation function for the combination, and are given below:

k	v(kT)	$\phi_{vv}(kT)$
0	0	0.2369704
1	0.09563301	0.2004822
2	0.2031604	0.1812449

3	0.2198356	0.1504768
4	0.1852145	0.1224713
5	0.1562003	0.09350676
6	0.1525930	0.06366732
7	0.1533819	0.03905807
8	0.1325538	0.02467862
9	-0.003237093	0.01176261
10	0.1229974	0.00000

From these values, the α's and β's for the power-series controller are calculated, as defined in Equations (3.15) and (3.16). These are given below:

$\alpha_{11} = 46.73314$	$\alpha_{21} = 339.1764$	$\alpha_{31} = 2797.805$
$\alpha_{12} = 341.5754$	$\alpha_{22} = 2520.276$	$\alpha_{32} = 21012.72$
$\alpha_{13} = 2771.327$	$\alpha_{23} = 21012.72$	$\alpha_{33} = 172522.6$
$\alpha_{14} = 23632.29$	$\alpha_{24} = 185515.6$	$\alpha_{34} = 1477812.0$
$\alpha_{15} = 207806.7$	$\alpha_{25} = 1687022.0$	$\alpha_{35} = 12998280.0$

$\alpha_{41} = 24492.86$	$\alpha_{51} = 221201.6$	$\beta_1 = 28.62194$
$\alpha_{42} = 185515.6$	$\alpha_{52} = 1687022.0$	$\beta_2 = 196.4032$
$\alpha_{43} = 1563037.0$	$\alpha_{53} = 13965200.0$	$\beta_3 = 1582.595$
$\alpha_{44} = 13138380.0$	$\alpha_{54} = 133686600.0$	$\beta_4 = 13717.33$
$\alpha_{45} = 115719200.0$	$\alpha_{55} = 1060478000.0$	$\beta_5 = 123298.6$

Finally, solving Equation (3.14) for the optimum coefficients of the power series, the following values are obtained:

$a_1 = 4.114004$
$a_2 = -0.8714087$
$a_3 = -0.000697533$
$a_4 = -0.00005014564$
$a_5 = 0.0006597933$

The mean-square error between the desired output and actual output for the combination of the power-series controller and the linear controller is given by

$$G^2 = \phi_{zz}(0) - \sum_{n=1}^{5} a_n \beta_n = 1.90 \tag{4.05}$$

For comparison, a power-series controller without storage may be calculated for the same plant and input signal. In this case the optimum coefficients are found to be

$a_1 = 105.5193$
$a_2 = -22.97353$
$a_3 = -0.01269078$
$a_4 = -0.000888152$
$a_5 = 0.01752794$

and the mean square error is found to be 2.15.

CONCLUSION

This example shows that a nonlinear controller with storage gives a performance much better than that of a linear controller, or a nonlinear controller with storage. As the calculation of nonlinear storage systems of higher order is more involved, this work has been limited to considering nonlinear systems of first order only. But, even with these, considerable improvements in performance of the controller is obtained. It is assumed that the computations can be carried out in a very short time, and the parameters adjusted immediately afterwards. In the ideal case, all this should not take more than one sampling interval, but the technique would still be valid if the statistics of the input and the plant dynamics vary slowly with time.

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